

JETS VIA HASSE-SCHMIDT DERIVATIONS

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ABSTRACT. This paper is intended to provide a general reference for jet spaces and jet differentials, valid in maximal generality (at the level of EGA). The approach is rather concrete, using Hasse-Schmidt (divided) higher differentials. Discussion of projectivized jet spaces (as in Green and Griffiths [G-G]) is included.

This paper contains a few brief notes on how to define jets using Hasse-Schmidt higher derivations.

I wrote them in order to better understand jets, and also to generalize the situation more fully to the situation of arbitrary schemes. This includes allowing singularities, working in arbitrary characteristic (including mixed characteristic), and working in the relative situation of one scheme over another.

Throughout this note, all rings (and algebras) are assumed to be commutative.

Most of this note consists of straightforward generalizations of the theory of derivations of order 1, as found in Grothendieck [EGA] or Matsumura [M]. Most of this is known already to the experts, but I am not aware of any general references, other than ([B-L-R], Sect. 9.6, proof of Lemma 2), [L], and ([Bu], Ch. 3, (3.14)). See also Green and Griffiths [G-G], in the context of complex manifolds.

Section 1 gives the basic definition of higher-order divided derivations and differentials, leading up to the basic property of jets that they correspond to arcs in the scheme. The fundamental object is the algebra $\mathrm{HS}_{B/A}^m$ of divided differentials, replacing $\bigoplus_{d \geq 0} S^d \Omega_{B/A}$; here $m \in \mathbb{N} \cup \{\infty\}$. The algebra $\mathrm{HS}_{B/A}^m$ is a graded algebra in which the higher differentials have varying degrees; therefore there is no obvious candidate for a module to replace $\Omega_{B/A}$.

Section 2 extends the usual first and second fundamental exact sequences to the context of higher differentials (with some changes in the leftmost term in each case). Section 3 shows that higher differentials are preserved under passing to étale covers; this then implies that the definitions are preserved under localization (which can also

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be shown directly). In Section 4 this fact is used to patch together the algebras $\mathrm{HS}_{B/A}^m$ to get a graded sheaf of \mathcal{O}_X -algebras $\mathrm{HS}_{X/Y}^m$ for an arbitrary morphism $X \rightarrow Y$ of schemes. The relative jet space $J_m(X/Y)$ can then be defined as $\mathbf{Spec} \mathrm{HS}_{X/Y}^m$. In particular, $J_m(X/Y)$ represents the functor $Z \mapsto \mathrm{Hom}_Y(Z[[t]]/(t^{m+1}), X)$ from Y -schemes Z to the category of sets (see Theorem 4.5 for notation). If $Y = Z = \mathrm{Spec} k$ for a field k , then this means that k -rational points on the jet space correspond to truncated arcs $\mathrm{Spec} k[t]/(t^{m+1}) \rightarrow X$ over k .

Basic properties of this jet space are described in Section 5. Section 6 shows that projectivized jet spaces (as in Green and Griffiths [G-G]) can be defined as $\mathbf{Proj} \mathrm{HS}_{X/Y}^m$ for $m > 0$. Finally, Section 7 discusses how one might define log jet spaces, but any actual work in this direction is postponed to a subsequent paper.

The Green-Griffiths projective jet spaces correspond to a quotient of the subset of “nonzero” jets by the action of \mathbb{G}_m corresponding to linear contractions and expansions of the corresponding arcs. Using the full automorphism group of $k[t]/(t^{m+1})$ or $k[[t]]$ gives a smaller quotient with a natural completion first defined by J. G. Semple [S] and further refined by J.-P. Demailly [De] and others. These jet spaces may be more suitable for arithmetic applications, since they throw out certain information which may be extraneous. These quotients will be explored in a subsequent paper.

It should be noted that, in some fields, the term “jet space” has a different meaning, namely the space parametrizing elements of the completed local ring $\widehat{\mathcal{O}}_{X,x}$, where X is a smooth scheme over a field k and x is a point of X . This type of jet space can be readily defined via Grothendieck’s theory of principal parts ([EGA 0], 20.4.14). See also [Y].

A more general type of (arc) jet space is described in a paper of A. Weil [W], in which he replaces $k[t]/(t^{m+1})$ with an Artinian local ring, in the context of real manifolds. I thank S. Kobayashi for mentioning this to me.

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§1. Basic definitions

We begin with a definition.

Definition 1.1. Let A be a ring, let $f: A \rightarrow B$ and $A \rightarrow R$ be A -algebras, and let $m \in \mathbb{N} \cup \{\infty\}$. Then a **higher derivation of order m from B to R over A** is a sequence (D_0, \dots, D_m) (or (D_0, D_1, \dots) if $m = \infty$), where $D_0: B \rightarrow R$ is an A -algebra homomorphism and $D_1, \dots, D_m: B \rightarrow R$ (or D_1, D_2, \dots if $m = \infty$) are homomorphisms of (additive) abelian groups, such that:

- (i). $D_i(f(a)) = 0$ for all $a \in A$ and all $i = 1, \dots, m$; and
- (ii). (Leibniz rule) for all $x, y \in B$ and all $k = 0, \dots, m$,

$$(1.1.1) \quad D_k(xy) = \sum_{i+j=k} D_i(x)D_j(y) .$$

Remarks. (a). We will often omit mention of m and A if they are clear from the context. Also, since there will never be a D_∞ , we will write (D_0, \dots, D_m) when $m = \infty$ to mean (D_0, D_1, \dots) , by abuse of notation.

(b). Instead of condition (i) above, Matsumura ([M], §27) assumes that the D_i are all A -module homomorphisms. This is an equivalent definition. Indeed, (i) and (ii) imply that the D_i are A -module homomorphisms. To see the converse it suffices to show by A -linearity that $D_i(1) = 0$ for all $i > 0$; this holds by applying the Leibniz rule to $1 = 1 \cdot 1$.

(c). Higher derivations were introduced by Hasse and Schmidt [H-S], and are often called *Hasse-Schmidt derivations*.

Example. If R has characteristic 0, if $R \subseteq B$, and if D is a derivation from B to R over A (in the usual sense), then

$$D_i := \frac{1}{i!} D^i, \quad i = 0, \dots, m$$

give a higher derivation of order m from B to R .

Remark 1.2. If $\phi: R \rightarrow R'$ is a homomorphism of A -algebras and (D_0, \dots, D_m) is a higher derivation from B to R , then $(\phi \circ D_0, \dots, \phi \circ D_m)$ is a higher derivation from B to R' . Thus, if $\text{Der}_A^m(B, R)$ denotes the set of higher derivations of order m from B to R , then $\text{Der}_A^m(B, \cdot)$ is a covariant functor from the category of A -algebras to the category of sets.

Definition 1.3. Let $f: A \rightarrow B$ and m be as above. Define a B -algebra $\text{HS}_{B/A}^m$ to be the quotient of the polynomial algebra

$$B[x^{(i)}]_{x \in B, i=1, \dots, m}$$

by the ideal I generated by the union of the sets

$$(1.3.1a) \quad \{(x+y)^{(i)} - x^{(i)} - y^{(i)} : x, y \in B; i = 1, \dots, m\},$$

$$(1.3.1b) \quad \{f(a)^{(i)} : a \in A; i = 1, \dots, m\}, \quad \text{and}$$

$$(1.3.1c) \quad \left\{ (xy)^{(k)} - \sum_{i+j=k} x^{(i)} y^{(j)} : x, y \in B; k = 0, \dots, m \right\},$$

where we identify $x^{(0)}$ with x for all $x \in B$ (and interpret $i = 1, \dots, m$ to mean $i \in \mathbb{Z}_{>0}$ when $m = \infty$, by the usual abuse of notation). We also define the **universal derivation** (d_0, \dots, d_m) from B to $\text{HS}_{B/A}^m$ by

$$d_i(x) = x^{(i)} \pmod{I}.$$

The resulting algebra $\text{HS}_{B/A}^m$ is an algebra over B ; it can also be viewed as an algebra over A via f . It is also a graded algebra (either over B or over A) in which the degree of $d_i x$ is i .

As examples, we have

$$\mathrm{HS}_{B/A}^0 = B$$

and

$$(1.4) \quad \mathrm{HS}_{B/A}^1 = \bigoplus_{d \geq 0} S^d \Omega_{B/A}$$

(note that $\mathrm{HS}_{B/A}^m$ is an algebra over B , not a module).

Remark 1.5. For $0 \leq i \leq j \leq \infty$, we have natural graded B -algebra homomorphisms $f_{ij}: \mathrm{HS}_{B/A}^i \rightarrow \mathrm{HS}_{B/A}^j$. These satisfy

$$f_{ik} = f_{jk} \circ f_{ij} \quad \text{for all } 0 \leq i \leq j \leq k \leq \infty$$

and

$$f_{ii} = \mathrm{Id} \quad \text{for all } i \in \mathbb{N} \cup \{\infty\}.$$

Thus, they form a directed system, and

$$(1.5.1) \quad \mathrm{HS}_{B/A}^\infty = \varinjlim_{m \in \mathbb{N}} \mathrm{HS}_{B/A}^m.$$

Proposition 1.6. *Let $A \rightarrow B$ and $A \rightarrow R$ be A -algebras, and let $m \in \mathbb{N} \cup \{\infty\}$. Given a derivation (D_0, \dots, D_m) from B to R , there exists a unique A -algebra homomorphism $\phi: \mathrm{HS}_{B/A}^m \rightarrow R$ such that*

$$(1.6.1) \quad (D_0, \dots, D_m) = (\phi \circ d_0, \dots, \phi \circ d_m).$$

Consequently $\mathrm{HS}_{B/A}^m$ (together with the universal derivation) represents the functor $\mathrm{Der}_A^m(B, \cdot)$.

Proof. Define

$$\phi_0: B[x^{(i)}]_{x \in B, i=1, \dots, m} \rightarrow R$$

by $x^{(i)} \mapsto D_i(x)$ for all $x \in B$, $i = 0, \dots, m$, where again $x^{(0)}$ means x . The properties of a derivation imply that the kernel of ϕ_0 contains the ideal (1.3.1), so we get a map ϕ satisfying (1.6.1). Moreover, (1.6.1) forces the choice of ϕ_0 , so ϕ is unique. Thus the map

$$(1.6.2) \quad \mathrm{Hom}_A(\mathrm{HS}_{B/A}^m, R) \rightarrow \mathrm{Der}_A^m(B, R)$$

defined by the right-hand side of (1.6.1) is bijective, so the second assertion holds. \square

To relate all of the above to jets, we have the following fact.

Lemma 1.7. *Let $f: A \rightarrow B$, R , and m be as above, and let*

$$(D_0, \dots, D_m) \in \text{Der}_A^m(B, R) .$$

Define $\phi: B \rightarrow R[t]/(t^{m+1})$ (if $m < \infty$) or $\phi: B \rightarrow R[[t]]$ (if $m = \infty$) by

$$(1.7.1) \quad x \mapsto D_0(x) + D_1(x)t + \dots + D_m(x)t^m \pmod{(t^{m+1})} .$$

Then ϕ lies in $\text{Hom}_A(B, R[t]/(t^{m+1}))$ or $\text{Hom}_A(B, R[[t]])$, respectively, and the resulting map

$$(1.7.2) \quad \text{Der}_A^m(B, R) \rightarrow \begin{cases} \text{Hom}_A(B, R[t]/(t^{m+1})), & m < \infty; \\ \text{Hom}_A(B, R[[t]]), & m = \infty \end{cases}$$

is bijective.

Proof. Since the D_i are all homomorphisms of the underlying additive groups, so is ϕ . The Leibniz rule implies that ϕ is multiplicative, so ϕ is a ring homomorphism. Finally, the conditions that $D_i(f(a)) = 0$ for all $a \in A$ and all $i > 0$ and that D_0 is an A -algebra homomorphism imply that ϕ is also a homomorphism of A -algebras. This proves the first assertion.

The map (1.7.2) is obviously injective, and surjectivity follows by reversing the steps in the above paragraph to show that given a map ϕ , (1.7.1) defines a higher derivation. Thus the map (1.7.2) is bijective. \square

As an immediate consequence, we have:

Corollary 1.8 (Jet desideratum; see ([G], Cor. 1 of Prop. 4)). *Let $A \rightarrow B$, $A \rightarrow R$, and m be as in Proposition 1.6. The bijections (1.6.2) and (1.7.2) define a natural bijection*

$$(1.8.1) \quad \text{Hom}_A(\text{HS}_{B/A}^m, R) \rightarrow \begin{cases} \text{Hom}_A(B, R[t]/(t^{m+1})), & m < \infty; \\ \text{Hom}_A(B, R[[t]]), & m = \infty \end{cases}$$

in which a map $\phi \in \text{Hom}_A(\text{HS}_{B/A}^m, R)$ is associated to the map $B \rightarrow R[t]/(t^{m+1})$ or $B \rightarrow R[[t]]$ given by

$$(1.8.2) \quad x \mapsto \phi(d_0x) + \phi(d_1x)t + \dots + \phi(d_mx)t^m \pmod{(t^{m+1})} .$$

Remark 1.9. From now on, we will write $R[[t]]/(t^{m+1})$ instead of $R[t]/(t^{m+1})$ (for $m < \infty$) or $R[[t]]$ (for $m = \infty$).

Remark 1.10. Suppose further that R is an algebra over B ; say $g: B \rightarrow R$. Then the bijection (1.8.1) takes $\text{Hom}_B(\text{HS}_{B/A}^m, R)$ to

$$\{\phi \in \text{Hom}_A(B, R[[t]]/(t^{m+1})) : z \circ \phi = g\} ,$$

where $z: R[[t]]/(t^{m+1}) \rightarrow R$ is the map inducing the identity on R and taking t to 0.

§2. Fundamental Exact Sequences

The fundamental exact sequences carry over almost directly from the case of differentials of order 1; the only differences stem from the fact that we are now working with algebras instead of modules.

Theorem 2.1 (First fundamental exact sequence). *Let $A \rightarrow B \rightarrow C$ be a sequence of ring homomorphisms, and let $m \in \mathbb{N} \cup \{\infty\}$. Then the sequence*

$$0 \rightarrow (\mathrm{HS}_{B/A}^m)^+ \mathrm{HS}_{C/A}^m \rightarrow \mathrm{HS}_{C/A}^m \rightarrow \mathrm{HS}_{C/B}^m \rightarrow 0$$

is an exact sequence of graded C -algebras; here $(\mathrm{HS}_{B/A}^m)^+$ denotes the ideal of elements of degree > 0 in the graded algebra $\mathrm{HS}_{B/A}^m$.

Proof. Since the term on the left is an ideal in $\mathrm{HS}_{C/A}^m$, the sequence is exact on the left.

The two terms $\mathrm{HS}_{C/A}^m$ and $\mathrm{HS}_{C/B}^m$ on the right are both quotients of the same polynomial algebra, and the generating set (1.3.1) for the ideal used to define $\mathrm{HS}_{C/A}^m$ is a subset of the corresponding generating set for $\mathrm{HS}_{C/B}^m$, so the map on the right is well defined and surjective. The difference of the two generating sets forms a generating set for the ideal $(\mathrm{HS}_{B/A}^m)^+$ in $\mathrm{HS}_{B/A}^m$, so the sequence is exact in the middle. \square

Note that the kernel of the map $\mathrm{HS}_{C/A}^m \rightarrow \mathrm{HS}_{C/B}^m$ contains elements such as $d_1 b \cdot d_1 c$ with $b \in B$ and $c \in C$, so it would not be correct to replace the term on the left with $(\mathrm{HS}_{B/A}^m)^+ \otimes_B C$.

In the case of order-1 differentials, the second fundamental exact sequence is a refinement of the first, via the fact that $\Omega_{C/B} = 0$ when $B \rightarrow C$ is surjective. In the present case, though, $\mathrm{HS}_{C/B}^m = C$, suggesting a second exact sequence of the form

$$? \rightarrow (\mathrm{HS}_{B/A}^m)^+ \mathrm{HS}_{C/A}^m \rightarrow \mathrm{HS}_{C/A}^m \rightarrow C \rightarrow 0.$$

Instead, though, taking the grading into account allows us to move the C two places to the left (see also Lemma 3.4):

Theorem 2.2 (Second fundamental exact sequence). *Let $A \rightarrow B \rightarrow C$ and m be as above. Assume also that $B \rightarrow C$ is surjective, and let I be its kernel. Let J be the ideal in $\mathrm{HS}_{B/A}^m$,*

$$(2.2.1) \quad J := (d_i x)_{i=0, \dots, m, x \in I}.$$

Then the sequence

$$(2.2.2) \quad 0 \rightarrow J \rightarrow \mathrm{HS}_{B/A}^m \rightarrow \mathrm{HS}_{C/A}^m \rightarrow 0$$

is an exact sequence of graded B -modules.

Proof. Exactness on the left is obvious. It is also obvious from the definitions that the natural map $\alpha: \text{HS}_{B/A}^m \rightarrow \text{HS}_{C/A}^m$ is surjective and that its kernel contains J .

From the definition of HS^m , we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & B[x^{(i)}]_{x \in B, i=1, \dots, m} & \longrightarrow & \text{HS}_{B/A}^m \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \alpha \\ 0 & \longrightarrow & K' & \longrightarrow & C[y^{(i)}]_{y \in C, i=1, \dots, m} & \longrightarrow & \text{HS}_{C/A}^m \longrightarrow 0 \end{array}$$

By (1.3.1), the leftmost vertical arrow is surjective, so by the Snake Lemma, the kernel of the middle vertical arrow maps onto $\text{Ker } \alpha$. But the kernel of the middle vertical arrow is generated by

$$I \cup \{d_i x - d_i y : i = 1, \dots, m; x, y \in B; x - y \in I\}.$$

This implies that the kernel of α is generated by the set $\{d_i x : i = 0, \dots, m, x \in I\}$, as in (2.2.1). \square

Remark 2.3. In (2.2.1), it suffices to let x vary over a generating set of I .

§3. Formally Étale Algebras

We recall the following definition:

Definition 3.1. Let C be an algebra over B . Then:

- (a). C is **formally unramified** over B if, for each B -algebra homomorphism $p: D \rightarrow E$ with nilpotent kernel, and for each B -algebra homomorphism $v: C \rightarrow E$, there exists *at most one* B -algebra homomorphism $u: C \rightarrow D$ such that $p \circ u = v$:

$$\begin{array}{ccc} C & \xrightarrow{v} & E \\ \uparrow & \searrow u & \uparrow p \\ B & \longrightarrow & D \end{array}$$

- (b). C is **formally smooth** over B if, for each surjective B -algebra homomorphism $p: D \rightarrow E$ with nilpotent kernel, and for each B -algebra homomorphism $v: C \rightarrow E$, there exists (at least one) B -algebra homomorphism $u: C \rightarrow D$ such that $p \circ u = v$.
- (c). C is **formally étale** over B if it is formally unramified and formally smooth over B .

Remark 3.2. Let $(D_i)_{i \in I}$ be a directed system of B -algebras with maps $(f_{ij})_{i, j \in I}$, let $(p_i)_{i \in I}$ be a system of B -algebra homomorphisms $p_i: D_i \rightarrow E$ compatible with the maps f_{ij} , let $D = \varinjlim_{i \in I} D_i$, and let $p: D \rightarrow E$ be the resulting map. If C is formally unramified over B and if the p_i all have nilpotent kernels, then the condition

in Definition 3.1a applies also to this map $p: D \rightarrow E$ (i.e., there is at most one $u: C \rightarrow D$ such that the above diagram commutes).

Similarly, if C is formally smooth over B , if the p_i are all surjective and have nilpotent kernels, and if the same holds for all the f_{ij} , then the condition in Definition 3.1b holds for $p: D \rightarrow E$ in this case.

In particular, these conditions apply to the map $z: R[[t]] \rightarrow R[[t]]/(t) = R$, so the proofs of Lemmas 3.3 and 3.5 are valid also when $m = \infty$.

Lemma 3.3. *If C is formally unramified over B , then $\mathrm{HS}_{C/B}^m = C$ for all $m \in \mathbb{N} \cup \{\infty\}$.*

Proof. We have the following isomorphisms:

$$\mathrm{Hom}_B(\mathrm{HS}_{C/B}^m, R) \xrightarrow{\sim} \mathrm{Hom}_B(C, R[[t]]/(t^{m+1})) \xrightarrow{\sim} \mathrm{Hom}_B(C, R)$$

for all B -algebras R . Indeed, the first arrow is Corollary 1.8. The second arrow is defined by composing with the morphism $z: R[[t]]/(t^{m+1}) \rightarrow R$ given by $t \mapsto 0$; it is easily seen to be surjective; and injectivity follows by applying the definition of formal unramifiedness to z , as in (3.3.1), below. Since this bijection holds for all R , we obtain $\mathrm{HS}_{C/B}^m = C$.

$$(3.3.1) \quad \begin{array}{ccc} C & \longrightarrow & R \\ \uparrow & \searrow & \uparrow z \\ B & \longrightarrow & R[[t]]/(t^{m+1}) \end{array} \quad \square$$

Lemma 3.4. *Let $A \rightarrow B \rightarrow C$ and m be as in Theorem 2.1. If $\mathrm{HS}_{C/B}^m = C$, then the natural map $\mathrm{HS}_{B/A}^m \otimes_B C \rightarrow \mathrm{HS}_{C/A}^m$ is surjective.*

Proof. This is a graded homomorphism of graded C -algebras, so it suffices to prove surjectivity on each graded part, by induction on the degree k . If $k = 0$, then the map is $C \rightarrow C$, which is clearly surjective. Let $k > 0$, and assume that surjectivity has been proved in all degrees $< k$. Then, by the first fundamental exact sequence, every homogeneous element of degree k of $\mathrm{HS}_{C/A}^m$ can be written as a sum of terms $\alpha \cdot \beta$, in which $\alpha \in \mathrm{HS}_{C/A}^m$ and $\beta \in (\mathrm{HS}_{B/A}^m)^+$. Moreover, we may assume that β is of the form $d_i x$ for some $i > 0$ and $x \in B$. But then α has degree $k - i < k$, so it lies in the image of $\mathrm{HS}_{B/A}^m \otimes_B C$ by induction. Thus the map is surjective in degree k , as was to be shown. \square

Lemma 3.5. *Let $A \rightarrow B \rightarrow C$ and m be as in Theorem 2.1, and further assume that C is formally smooth over B . Then the natural map*

$$\alpha: \mathrm{HS}_{B/A}^m \otimes_B C \rightarrow \mathrm{HS}_{C/A}^m$$

is invertible on the left; i.e., there exists a C -algebra homomorphism

$$\beta: \mathrm{HS}_{C/A}^m \rightarrow \mathrm{HS}_{B/A}^m \otimes_B C$$

such that $\beta \circ \alpha = \text{Id}_{\text{HS}_{B/A}^m \otimes_B C}$.

Proof. We first claim that, for any C -algebra R , the map

$$\text{Hom}_C(\text{HS}_{C/A}^m, R) \rightarrow \text{Hom}_C(\text{HS}_{B/A}^m \otimes_B C, R)$$

is surjective. Indeed, we have $\text{Hom}_C(\text{HS}_{B/A}^m \otimes_B C, R) = \text{Hom}_B(\text{HS}_{B/A}^m, R)$; this together with Remark 1.10 implies that the above map is equivalent to the map

$$(3.5.1) \quad \begin{aligned} & \{\phi \in \text{Hom}_A(C, R[[t]]/(t^{m+1})) : z \circ \phi = g\} \\ & \rightarrow \{\phi \in \text{Hom}_A(B, R[[t]]/(t^{m+1})) : z \circ \phi = f\}, \end{aligned}$$

where $f: B \rightarrow R$ and $g: C \rightarrow R$ express the B -algebra and C -algebra structures on R , respectively. Surjectivity of (3.5.1) then follows from the definition of formal smoothness, as in (3.3.1).

The lemma then follows by applying this claim to $R = \text{HS}_{B/A}^m \otimes_B C$. \square

Theorem 3.6. *Let $A \rightarrow B \rightarrow C$ be a sequence of ring homomorphisms, and let $m \in \mathbb{N} \cup \{\infty\}$. If C is formally étale over B , then the natural map*

$$\text{HS}_{B/A}^m \otimes_B C \rightarrow \text{HS}_{C/A}^m$$

is an isomorphism of graded C -algebras.

Proof. It is surjective by Lemma 3.3 and injective by Lemma 3.5. \square

§4. Jet Schemes

Two localization properties of the algebras $\text{HS}_{B/A}^m$ allow one to define $\text{HS}_{X/Y}^m$ for an arbitrary morphism of schemes $X \rightarrow Y$.

Lemma 4.1. *Let $f: A \rightarrow B$ be a ring homomorphism, and let $m \in \mathbb{N} \cup \{\infty\}$. Let S be a multiplicative subset of A . If f factors through the canonical morphism $A \rightarrow S^{-1}A$, then the identity map on $B[x^{(i)}]_{x \in B, i=1, \dots, m}$ induces an isomorphism*

$$\text{HS}_{B/A}^m \xrightarrow{\sim} \text{HS}_{B/S^{-1}A}^m.$$

Proof. Let $f': S^{-1}A \rightarrow B$ be the factored homomorphism. The only difference between the ideals (1.3.1) for B/A and for $B/S^{-1}A$ comes from (1.3.1b), since the image of f' may be larger than the image of f . But, for all $a \in A$, $s \in S$, and $k \in \{1, \dots, m\}$, we have

$$0 = d_k f(a) = d_k(f(s)f'(s^{-1}a)) = \sum_{i+j=k} d_i f(s) \cdot d_j f'(s^{-1}a) = f(s) d_k f'(s^{-1}a)$$

in $\mathrm{HS}_{B/A}^m$, by induction on k , so since $f(s)$ is invertible in B , $d_k f'(s^{-1}a)$ vanishes in $\mathrm{HS}_{B/A}^m$. Thus the two ideals coincide. \square

Lemma 4.2. *Let $f: A \rightarrow B$ and m be as in Lemma 4.1, and let S be a multiplicative subset of B . Then the obvious map*

$$B[x^{(i)}]_{x \in B, i=1, \dots, m} \rightarrow S^{-1}B[y^{(i)}]_{y \in S^{-1}B, i=1, \dots, m}$$

induces an isomorphism

$$S^{-1}\mathrm{HS}_{B/A}^m \xrightarrow{\sim} \mathrm{HS}_{S^{-1}B/A}^m.$$

Proof. This is immediate from Theorem 3.6, since localization algebras are formally étale. \square

These two lemmas now allow us to define a sheaf of \mathcal{O}_X -algebras for any scheme morphism $X \rightarrow Y$ (not necessarily separated).

Theorem 4.3. *Let $f: X \rightarrow Y$ be a morphism of schemes, and let $m \in \mathbb{N} \cup \{\infty\}$. Then there exists a quasi-coherent sheaf $\mathrm{HS}_{X/Y}^m$ of \mathcal{O}_X -algebras such that (i) for each pair of open affines $\mathrm{Spec} A \subset Y$ and $\mathrm{Spec} B \subseteq f^{-1}(\mathrm{Spec} A)$, there exists an isomorphism*

$$\phi_{B/A}: \Gamma(\mathrm{Spec} B, \mathrm{HS}_{X/Y}^m) \xrightarrow{\sim} \mathrm{HS}_{B/A}^m$$

of B -algebras, and (ii) the various $\phi_{B/A}$ are compatible with the localization isomorphisms of Lemmas 4.1 and 4.2. Moreover, the collection $(\mathrm{HS}_{X/Y}^m, (\phi_{B/A})_{A,B})$ is unique up to unique isomorphism.

Proof. One can construct a suitable sheaf $\mathrm{HS}_{X/Y}^m$ by first constructing it in the special case when Y is affine, using Lemma 4.2 and the fact that the localization map of that lemma is functorial in S . Then generalize to arbitrary Y using both of the above lemmas, the fact that their localization maps are functorial in the respective multiplicative subsets, and the fact that the diagram

$$\begin{array}{ccc} T^{-1}\mathrm{HS}_{B/A}^m & \longrightarrow & T^{-1}\mathrm{HS}_{B/S^{-1}A}^m \\ \downarrow & & \downarrow \\ \mathrm{HS}_{T^{-1}B/A}^m & \longrightarrow & \mathrm{HS}_{T^{-1}B/S^{-1}A}^m \end{array}$$

commutes for all $A \rightarrow B$ and all multiplicative subsets S of A and T of B such that $A \rightarrow B$ extends to $S^{-1}A \rightarrow B$.

The existence of the $\phi_{B/A}$ follows from the above glueing process, provided that one glues over the set of all open affines at each stage. The uniqueness assertion follows similarly. \square

Definition 4.4. Let $X \rightarrow Y$ and m be as above. Then the **scheme of m -jet differentials** of X over Y is the scheme

$$J_m(X/Y) := \mathbf{Spec} \, \mathrm{HS}_{X/Y}^m .$$

If $A \rightarrow B$ is a ring homomorphism, then we also write

$$J_m(B/A) = J_m(\mathrm{Spec} \, B / \mathrm{Spec} \, A)$$

(which is equal to $\mathrm{Spec} \, \mathrm{HS}_{B/A}^m$).

Theorem 4.5 ([K-I], Prop. 2.9). *Let $X \rightarrow Y$ and m be as above. Then the scheme $J_m(X/Y)$ represents the functor from Y -schemes to sets, given by*

$$(4.5.1) \quad Z \mapsto \mathrm{Hom}_Y(Z \times_{\mathbb{Z}} \mathrm{Spec} \, \mathbb{Z}[t]/(t^{m+1}), X)$$

if m is finite, or

$$(4.5.2) \quad Z \mapsto \mathrm{Hom}_Y(Z \times_{\mathbb{Z}} \mathrm{Spf} \, \mathbb{Z}[[t]], X) ,$$

if $m = \infty$.

Proof. First, for brevity, we let $Z[[t]]/(t^{m+1})$ denote the scheme $Z \times_{\mathbb{Z}} \mathrm{Spec} \, \mathbb{Z}[t]/(t^{m+1})$ if m is finite, or the formal scheme $Z \times_{\mathbb{Z}} \mathrm{Spf} \, \mathbb{Z}[[t]]$ if $m = \infty$. We then need to construct a bijection

$$(4.5.3) \quad \mathrm{Hom}_Y(Z[[t]]/(t^{m+1}), X) \rightarrow \mathrm{Hom}_Y(Z, J_m(X/Y)) .$$

Since the category of formal schemes contains the category of schemes as a full subcategory ([EGA], I 10.4.8), we will refer to $Z[[t]]/(t^{m+1})$ as a formal scheme even in the case in which m is finite. Also, we note that if m is finite and R is a ring with discrete topology, then the topology of $R[[t]]/(t^{m+1}) = R[t]/(t^{m+1})$ is also discrete, so $\mathrm{Spf} \, R[t]/(t^{m+1}) = \mathrm{Spec} \, R[t]/(t^{m+1})$. On the other hand, when $m = \infty$, we note that the product is taken in the category of formal schemes, so for example $(\mathrm{Spec} \, A) \times_{\mathbb{Z}} \mathbb{Z}[[t]] = \mathrm{Spf} \, A[[t]]$, by ([EGA], I 10.7.2).

The continuous ring homomorphism $\mathbb{Z}[[t]]/(t^{m+1}) \rightarrow \mathbb{Z}$ given by $t \mapsto 0$ gives a morphism $Z \rightarrow Z[[t]]/(t^{m+1})$, and therefore a map

$$\mathrm{Hom}_Y(Z[[t]]/(t^{m+1}), X) \rightarrow \mathrm{Hom}_Y(Z, X) .$$

We also have a map $\mathrm{Hom}_Y(Z, J_m(X/Y)) \rightarrow \mathrm{Hom}_Y(Z, X)$, obtained by composing with the structural map $J_m(X/Y) \rightarrow X$. The bijection (4.5.3) will be constructed so as to preserve fibers of the maps to $\mathrm{Hom}_Y(Z, X)$.

First consider the case in which X , Y , and Z are affine, say $X = \mathrm{Spec} \, B$, $Y = \mathrm{Spec} \, A$, and $Z = \mathrm{Spec} \, R$. We work in the category of formal schemes, so A ,

B , and R will have the discrete topology and $\text{Spec } B = \text{Spf } B$, etc. Then the bijection (4.5.3) follows immediately from Corollary 1.8. Indeed, by ([EGA], I 10.4.6 and I 10.1.3), we have $\text{Hom}_Y(Z[[t]]/(t^{m+1}), X) = \text{Hom}_A(B, R[[t]]/(t^{m+1}))$. That (4.5.3) preserves fibers of the maps to $\text{Hom}_Y(Z, X)$ in this case follows by looking at the constant term in (1.8.2).

The general case of the theorem now follows by a standard glueing argument, working over a fixed element of $\text{Hom}_Y(Z, X)$. \square

Proposition 4.6. *For all m , $J_m(X/Y)$ is affine, and therefore quasi-compact and separated, over X .*

Proof. Trivial. \square

Let $X \rightarrow Y$ be a morphism of schemes. Then the maps $f_{ij}: \text{HS}_{B/A}^i \rightarrow \text{HS}_{B/A}^j$ of Remark 1.5 give rise to graded homomorphisms

$$(4.7) \quad f_{ij}: \text{HS}_{X/Y}^i \rightarrow \text{HS}_{X/Y}^j$$

for all $0 \leq i \leq j \leq \infty$, which again form a directed system as in Remark 1.5.

This in turn translates into schemes: The f_{ij} give rise to morphisms

$$(4.8) \quad \pi_{ji}: J_j(X/Y) \rightarrow J_i(X/Y)$$

over X satisfying $\pi_{ki} = \pi_{ji} \circ \pi_{kj}$ for all $0 \leq i \leq j \leq k \leq \infty$ and $\pi_{ii} = \text{Id}$ for all $i \in \mathbb{N} \cup \{\infty\}$.

If $X \rightarrow Y$ is a morphism of schemes, then (1.5.1) gives

$$(4.9) \quad \text{HS}_{X/Y}^\infty = \varinjlim_{m \in \mathbb{N}} \text{HS}_{X/Y}^m$$

and

$$(4.10) \quad J_\infty(X/Y) = \varinjlim_{m \in \mathbb{N}} J_m(X/Y).$$

For all $m \in \mathbb{N} \cup \{\infty\}$, the natural graded map

$$\text{HS}_{B/A}^m \rightarrow \text{HS}_{B/B}^m = B$$

(as in the first fundamental exact sequence; it takes $d_j x$ to 0 for all $x \in B$ and all $j = 1, \dots, m$) gives rise to a map

$$(4.11) \quad \text{HS}_{X/Y}^m \rightarrow \mathcal{O}_X$$

of graded \mathcal{O}_X -algebras, which in turn gives rise to “zero sections”

$$(4.12) \quad s_m: X \rightarrow J_m(X/Y)$$

satisfying $\pi_m \circ s_m = \text{Id}_X$ for all m and $\pi_{ji} \circ s_j = s_i$ for all $0 \leq i \leq j \leq \infty$.

Example 4.13. Let k be a field of characteristic $\neq 2$, let $Y = \operatorname{Spec} k$, and let

$$X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$$

be the cuspidal cubic curve. Then

$$J_1(X/Y) = \operatorname{Spec} k[x, y, d_1x, d_1y]/(y^2 - x^3, 2y d_1y - 3x^2 d_1x)$$

and

$$\begin{aligned} J_2(X/Y) = \operatorname{Spec} k[x, y, d_1x, d_1y, d_2x, d_2y]/(y^2 - x^3, \\ 2y d_1y - 3x^2 d_1x, \\ (d_1y)^2 + 2y d_2y - 3x(d_1x)^2 - 3x^2 d_2x). \end{aligned}$$

Therefore the fiber of $\pi_1: J_1(X/Y) \rightarrow X$ over the point $(0, 0)$ is a copy of \mathbb{A}_k^2 , but the fiber of π_2 over the origin is only isomorphic to \mathbb{A}_k^3 , spanned by d_1x , d_2x , and d_2y (but not d_1y). In particular, the restriction $d_1y = 0$ in the fiber over the origin occurs only in J_2 and not in J_1 , so the map $f_{12}: \operatorname{HS}_{X/Y}^1 \rightarrow \operatorname{HS}_{X/Y}^2$ is not injective, and the map $\pi_{21}: J_2(X/Y) \rightarrow J_1(X/Y)$ is not surjective. See also Corollary 5.11.

Remark 4.14. Let $m \in \mathbb{N} \cup \{\infty\}$. Then $\operatorname{HS}_{B/A}^m$ is functorial in pairs $A \rightarrow B$, and $\operatorname{HS}_{X/Y}^m$ and $J_m(X/Y)$ are functorial in pairs $X \rightarrow Y$. Indeed, a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of rings and ring homomorphisms induces a functorial commutative diagram

$$(4.14.1) \quad \begin{array}{ccc} \operatorname{HS}_{B/A}^m & \xrightarrow{\operatorname{HS}_{\phi}^m} & \operatorname{HS}_{B'/A'}^m \\ \uparrow & & \uparrow \\ B & \xrightarrow{\phi} & B', \end{array}$$

taking $d_i b \in \operatorname{HS}_{B/A}^m$ to $d_i \phi(b) \in \operatorname{HS}_{B'/A'}^m$ for all $b \in B$ and all i . This in turn induces a B' -algebra homomorphism

$$(4.14.2) \quad \operatorname{HS}_{B/A}^m \otimes_B B' \rightarrow \operatorname{HS}_{B'/A'}^m,$$

again functorially (via transitivity of base change for tensor). Of particular note are the special cases $B' = B \otimes_A A'$ (base change in A) and $A = A'$ (functoriality in B).

Similarly, a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

of schemes induces a graded $\mathcal{O}_{X'}$ -algebra homomorphism

$$(4.14.3) \quad \mathrm{HS}_f^m : f_* \mathrm{HS}_{X/Y}^m \rightarrow \mathrm{HS}_{X'/Y'}^m$$

and a commutative diagram

$$(4.14.4) \quad \begin{array}{ccc} J_m(X/Y) & \xrightarrow{J_m(f)} & J_m(X'/Y') \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array},$$

functorially in both cases.

If f is a closed immersion then so is $J_m(f)$, by the two fundamental exact sequences (Theorems 2.1 and 2.2), or by direct computation. Similar statements hold if f is an affine morphism or a quasi-compact morphism.

If $Y = Y'$ and $Y \rightarrow Y'$ is the identity map, then we write $J_m(f)$ as $J_m(f/Y)$ instead. In this case, if f is an open immersion, then so is $J_m(f/Y)$. Consequently, if f is an immersion, then so is $J_m(f/Y)$.

See also Example 5.12 for some counterexamples along these lines.

§5. Functorial Properties of $\mathrm{HS}_{X/Y}^m$ and $J_m(X/Y)$

Polynomial Algebras

Proposition 5.1. *Let A be a ring, let B be the polynomial algebra $B = A[x_i]_{i \in I}$, and let $m \in \mathbb{N} \cup \{\infty\}$. Then $\mathrm{HS}_{B/A}^m$ is the polynomial algebra $B[d_j x_i]_{i \in I, j=1, \dots, m}$.*

Proof. By Corollary 1.8,

$$\begin{aligned} \mathrm{Hom}_A(\mathrm{HS}_{B/A}^m, R) &= \mathrm{Hom}_A(A[x_i]_{i \in I}, R[[t]]/(t^{m+1})) \\ &= \prod_{i \in I} R[[t]]/(t^{m+1}) \\ &= \prod_{\substack{i \in I \\ j=0, \dots, m}} R \\ &= \mathrm{Hom}_A(A[x_i^{(j)}]_{i \in I, j=0, \dots, m}, R) \end{aligned}$$

for all A -algebras R . Therefore $\mathrm{HS}_{B/A}^m \cong A[x_i^{(j)}]_{i \in I, j=0, \dots, m}$ (as an A -algebra). Tracking down the above maps leads to the fact that $x_i^{(j)}$ corresponds to $d_j x_i$ for all i and j , so $\mathrm{HS}_{B/A}^m \cong B[d_j x_i]_{i \in I, j=1, \dots, m}$, as a graded B -algebra. \square

Corollary 5.2. *For any scheme Y ,*

$$\mathrm{HS}_{\mathbb{A}_Y^n/Y}^m \cong \mathcal{O}_Y[x_1, \dots, x_n][d_j x_i]_{i=1, \dots, n, j=1, \dots, m}$$

and

$$J_m(\mathbb{A}_Y^n/Y) \cong \mathbb{A}_Y^n \times_Y \mathbb{A}_Y^{nm}.$$

Proof. Immediate. \square

Corollary 5.3. *Let B be an algebra over A , presented as*

$$B \cong A[x_i]_{i \in I} / (f_j)_{j \in J}.$$

Then

$$\mathrm{HS}_{B/A}^m \cong B[d_k x_i]_{i \in I, k=1, \dots, m} / (d_k f_j)_{j \in J, k=1, \dots, m}.$$

Proof. This follows from Proposition 5.1 and the second fundamental exact sequence (Theorem 2.2), via Remark 2.3. \square

Corollary 5.4. *Let $m \in \mathbb{N}$.*

- (a). *If Y is a scheme and X is locally of finite type over Y , then $J_m(X/Y)$ is of finite type over X .*
- (b). *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a commutative diagram of schemes. If f is of finite type (resp. locally of finite type), then so is $J_m(f): J_m(X/Y) \rightarrow J_m(X'/Y')$.

Proof. Obvious. \square

Base Change

Lemma 5.5. *Let B be an A -algebra, let $A \rightarrow A'$ be a ring homomorphism, let $B' = B \otimes_A A'$, and let $m \in \mathbb{N} \cup \{\infty\}$. Then the natural map (4.14.2)*

$$\mathrm{HS}_{B/A}^m \otimes_B B' \rightarrow \mathrm{HS}_{B'/A'}^m$$

defined by $d_i x \otimes b \mapsto b d_i(x \otimes 1)$ is an isomorphism of graded B' -algebras.

Proof. As an A' -algebra, the domain of the above map is the same as $\mathrm{HS}_{B/A}^m \otimes_{A'} A'$, and for all A' -algebras R , we have

$$\begin{aligned} \mathrm{Hom}_{A'}(\mathrm{HS}_{B/A}^m \otimes_{A'} A', R) &= \mathrm{Hom}_A(\mathrm{HS}_{B/A}^m, R) \\ &= \mathrm{Hom}_A(B, R[[t]]/(t^{m+1})) \\ &= \mathrm{Hom}_{A'}(B', R[[t]]/(t^{m+1})) \\ &= \mathrm{Hom}_{A'}(\mathrm{HS}_{B'/A'}^m, R) \end{aligned}$$

Therefore $\mathrm{HS}_{B/A}^m \otimes_B B' \cong \mathrm{HS}_{B'/A'}^m$ as A' -algebras. This isomorphism is the same as the map indicated in the statement of the lemma, so it is a B' -algebra isomorphism, and it preserves the grading. \square

(This can also be proved explicitly by Corollary 5.3.)

Translating this into the language of schemes gives the following immediate corollary:

Proposition 5.6. *Let $X \rightarrow Y$ be schemes, let $Y' \rightarrow Y$ be a morphism of schemes, let $X' = X \times_Y Y'$, let $p: X' \rightarrow X$ be the canonical projection, and let $m \in \mathbb{N} \cup \{\infty\}$. Then the homomorphism (4.14.3)*

$$(5.6.1) \quad p_* \mathrm{HS}_{X'/Y'}^m \rightarrow \mathrm{HS}_{X/Y}^m$$

is an isomorphism of graded \mathcal{O}_X -algebras, and the morphism

$$(5.6.2) \quad J_m(X'/Y') \rightarrow J_m(X/Y) \times_X X'$$

induced by (4.14.4) is an isomorphism of schemes over X' .

Products

In this subsection, tensor products will always be taken over all $i \in I$. Infinite tensor products are taken as in ([E], Prop. 16.5).

Lemma 5.7. *For each $i \in I$ let B_i be an A -algebra, let $B = \bigotimes_A B_i$, and let m be as above. Then the natural map*

$$\bigotimes_B (\mathrm{HS}_{B_i/A}^m \otimes_{B_i} B) \rightarrow \mathrm{HS}_{B/A}^m$$

is an isomorphism of graded B -algebras. Here the map satisfies

$$(5.7.1) \quad (\cdots \otimes 1 \otimes 1 \otimes (d_j x \otimes b) \otimes 1 \otimes 1 \otimes \cdots) \mapsto b d_j (\cdots \otimes 1 \otimes 1 \otimes x \otimes 1 \otimes 1 \otimes \cdots)$$

for all $b \in B$ and all $x \in B_i$, where in each expression x occurs in the i^{th} place.

Proof. As before, we first consider the corresponding A -algebra map

$$(5.7.2) \quad \bigotimes_A \mathrm{HS}_{B_i/A}^m \rightarrow \mathrm{HS}_{B/A}^m .$$

We have

$$\begin{aligned}
 \mathrm{Hom}_A\left(\bigotimes_A \mathrm{HS}_{B_i/A}^m, R\right) &= \prod_{i \in I} \mathrm{Hom}_A(\mathrm{HS}_{B_i/A}^m, R) \\
 &= \prod_{i \in I} \mathrm{Hom}_A(B_i, R[[t]]/(t^{m+1})) \\
 &= \mathrm{Hom}_A(B, R[[t]]/(t^{m+1})) \\
 &= \mathrm{Hom}_A(\mathrm{HS}_{B/A}^m, R)
 \end{aligned}$$

for all A -algebras R , so we have an isomorphism (5.7.2). We leave it to the reader to check that this isomorphism satisfies (5.7.1), and hence is an isomorphism of graded B -algebras. \square

(Again, this result can also be proved explicitly by Corollary 5.3.)

Translating this into schemes gives as an immediate corollary (in which, again, products are taken over all $i \in I$):

Proposition 5.8. *Let Y be a scheme, let X_i ($i \in I$) be Y -schemes, let $X = \prod_Y X_i$, and let m be as above. Then the natural map*

$$\bigotimes_{\mathcal{O}_X} (\mathrm{HS}_{X_i/Y}^m \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_X) = \bigotimes_{\mathcal{O}_Y} \mathrm{HS}_{X_i/Y}^m \rightarrow \mathrm{HS}_{X/Y}^m$$

is an isomorphism of graded \mathcal{O}_X -algebras, and

$$J_m(X/Y) \rightarrow \prod_Y J_m(X_i/Y) = \prod_X (J_m(X_i/Y) \times_{X_i} X)$$

is an isomorphism of X -schemes.

Étale Morphisms

Proposition 5.9. *Let Y be a scheme, let $f: X_1 \rightarrow X_2$ be an étale morphism of schemes over Y , and let $m \in \mathbb{N} \cup \{\infty\}$. Then the natural map*

$$f^* \mathrm{HS}_{X_2/Y}^m \rightarrow \mathrm{HS}_{X_1/Y}^m$$

is an isomorphism of graded \mathcal{O}_{X_1} -algebras, and also

$$J_m(X_1/Y) \cong J_m(X_2/Y) \times_{X_2} X_1$$

as schemes over X_1 . In particular, $J_m(f/Y)$ is also étale.

Proof. The first two assertions are immediate from Theorem 3.6. The last sentence follows from the second assertion and the fact that étaleness is preserved under base change. \square

Smooth Morphisms

Proposition 5.10. *Let $f: X \rightarrow Y$ be a smooth morphism and let $m \in \mathbb{N} \cup \{\infty\}$. Then X is covered by open sets U for which $J_m(U/Y) \cong \mathbb{A}_U^{dm}$, where $d \in \mathbb{N}$ is the relative dimension of U over Y . Moreover, the sets U may be chosen independent of m .*

Proof. By ([EGA], IV 17.11.4), smoothness of f implies that X is covered by open sets U for which there exist integers $n \geq 0$ and étale morphisms $g: U \rightarrow \mathbb{P}_Y^n$ such that $f|_U$ factors through g and the canonical projection $\mathbb{P}_Y^n \rightarrow Y$. (The converse also holds; this is a convenient criterion for smoothness.)

The result is then immediate from Corollary 5.2 and Proposition 5.9. \square

Corollary 5.11. *If $f: X \rightarrow Y$ is a smooth morphism and $0 \leq i \leq j \leq \infty$, then $f_{ij}: \mathrm{HS}_{X/Y}^i \rightarrow \mathrm{HS}_{X/Y}^j$ is injective and $\pi_{ji}: J_j(X/Y) \rightarrow J_i(X/Y)$ is surjective. (See Example 4.13 for a counterexample when f is not smooth.)*

Example 5.12. Let k be a field, let $X_1 = X_2 = \mathbb{P}_k^1$, let $f: X_1 \rightarrow X_2$ be the map given in affine coordinates by $x \mapsto x^e$, where $\mathrm{char} k \nmid e$, and let $m \in \mathbb{Z}_{>0} \cup \{\infty\}$. Then $J_m(f/k)$ is dominant but not surjective, hence not closed. (For $m = 1$ this can be computed directly; for $m > 1$, use Corollary 5.11.) Therefore the property of being closed, proper, or projective, are not preserved by passing from f to $J_m(f/k)$ ([I], 2.4).

Example 5.13. This example describes how a construction used in diophantine geometry can be phrased in terms of these jet spaces. Suppose $f: X \rightarrow Y$ is a smooth morphism of relative dimension n ; $\sigma: Y \rightarrow X$ is a section; E is its image; \mathcal{I} is the ideal sheaf of E (with reduced induced subscheme structure); U is some open subset of X meeting E ; and $b \in \mathcal{O}_X(U)$ is a function vanishing to order $\geq m$ along E . After possibly shrinking U , we may write $\mathcal{I} = (x_1, \dots, x_n)$ with $x_1, \dots, x_n \in \mathcal{O}_X(U)$. Since $b \in \mathcal{I}^m$, we may write

$$(5.13.1) \quad b = \sum_{\substack{\mathbf{i} \in \mathbb{N}^n \\ |\mathbf{i}| = m}} a_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n}$$

with $a_{\mathbf{i}} \in \mathcal{O}_X(U)$ for all \mathbf{i} . Now consider $(d_m b)|_E$. If we substitute the above expression for b , and apply Leibniz' rule repeatedly, we get

$$(d_n b)|_E = \sum a_{\mathbf{i}} (d_1 x_1)^{i_1} \cdots (d_1 x_n)^{i_n}$$

(all other terms involve some x_i without any d_j applied ($j > 0$), hence vanish when restricted to E). This expression may then be regarded as an element of $(S^m \Omega_{X/Y})|_E$. We need to check that it is well defined (in terms of the choice of coefficients $a_{\mathbf{i}}$ in (5.13.1)). This is because, by Corollary 5.11, the map

$$S^m \Omega_{X/Y} = (\mathrm{HS}_{X/Y}^1)_m \rightarrow (\mathrm{HS}_{X/Y}^m)_m$$

is injective. Since f is smooth, the above sheaves are all locally free, so the above map remains injective when restricted to E . Thus, the construction is well defined.

Similarly, if b is a local section of a line sheaf \mathcal{L} vanishing to order $\geq m$ along E , then the above construction gives a local section $(d_m b)|_E \in (\mathcal{L} \otimes S^m \Omega_{X/Y})|_E$.

§6. Green-Griffiths Projective Jet Spaces

Green and Griffiths [G-G] defined projectivized jet spaces by defining an action of \mathbb{G}_m on $J_m(X/Y) \setminus \text{Im } s_m$ and constructing a quotient space $P_m(X/Y)$. (They did the construction only for manifolds over \mathbb{C} , but the construction readily generalizes.) We repeat the construction here, in full generality.

Definition 6.1. Let $A \rightarrow B$ be a ring homomorphism and let $m \in \mathbb{N} \cup \{\infty\}$. By (1.3.1) there is a unique graded B -homomorphism

$$\phi_m: \text{HS}_{B/A}^m \rightarrow \text{HS}_{B/A}^m[z]$$

such that $d_j x \mapsto z^j d_j x$ for all $x \in B$ and all $j = 0, \dots, m$. (This will also be denoted ϕ when m is clear from the context.) If $X \rightarrow Y$ is a morphism of schemes, then the ϕ_m over affine pieces of X and Y glue together to give a unique homomorphism of graded \mathcal{O}_X -algebras

$$\phi_m: \text{HS}_{X/Y}^m \rightarrow \text{HS}_{X/Y}^m[z],$$

and hence a unique morphism

$$\psi_m: \mathbb{A}^1 \times J_m(X/Y) \rightarrow J_m(X/Y)$$

of schemes over X .

Intuitively, one thinks of ψ_m as follows. Assume that X and Y are affine, say $X = \text{Spec } B$ and $Y = \text{Spec } A$. Then a closed point in $J_m(X/Y)$ corresponds to a ring homomorphism $\gamma: B \rightarrow k[[t]]/(t^{m+1})$, and $\psi_m(z, \gamma)$ is obtained by composing γ with the map $k[[t]]/(t^{m+1}) \rightarrow k[[t]]/(t^{m+1})$ defined by $t \mapsto zt$. In other words, the germ of a curve is dilated by a factor of z .

This can be made rigorous (via the “jet desideratum”) as follows. For all A -algebras B and R , and all $z_0 \in A$, the diagram

$$(6.2) \quad \begin{array}{ccc} \text{Hom}_A(\text{HS}_{B/A}^m, R) & \longrightarrow & \text{Hom}_A(B, R[[t]]/(t^{m+1})) \\ \downarrow & & \downarrow \\ \text{Hom}_A(\text{HS}_{B/A}^m, R) & \longrightarrow & \text{Hom}_A(B, R[[t]]/(t^{m+1})) \end{array}$$

in which the vertical map on the left is composition with the map

$$\text{HS}_{B/A}^m \xrightarrow{\phi_m} \text{HS}_{B/A}^m[z] \xrightarrow{z \mapsto z_0} \text{HS}_{B/A}^m$$

and the vertical map on the right is composition with

$$R[[t]]/(t^{m+1}) \xrightarrow{t \mapsto z_0 t} R[[t]]/(t^{m+1}).$$

Proposition 6.3. *Let $X \rightarrow Y$ and m be as above. Then the map ϕ_m satisfies:*

(a). *The composite map*

$$J_m(X/Y) \xrightarrow{(0, \text{Id}_{J_m(X/Y)})} \mathbb{A}^1 \times J_m(X/Y) \xrightarrow{\psi_m} J_m(X/Y)$$

equals the map $s_m \circ \pi_m: J_m(X/Y) \rightarrow J_m(X/Y)$, where $\pi_m: J_m(X/Y) \rightarrow X$ is the structural morphism.

(b). *The composite map*

$$J_m(X/Y) \xrightarrow{(1, \text{Id}_{J_m(X/Y)})} \mathbb{A}^1 \times J_m(X/Y) \xrightarrow{\psi_m} J_m(X/Y)$$

equals the identity on $J_m(X/Y)$.

(c). *The maps ψ_m are compatible with the maps π_{ji} : the diagram*

$$\begin{array}{ccc} \mathbb{A}^1 \times J_j(X/Y) & \xrightarrow{\psi_j} & J_j(X/Y) \\ \downarrow \text{Id}_{\mathbb{A}^1} \times \pi_{ji} & & \downarrow \pi_{ji} \\ \mathbb{A}^1 \times J_i(X/Y) & \xrightarrow{\psi_i} & J_i(X/Y) \end{array}$$

commutes for all $0 \leq i \leq j \leq \infty$.

(d). *The restriction of ψ_m to $\mathbb{G}_m \times (J_m(X/Y) \setminus \text{Im } s_m)$ defines a group action, which is faithful on the set of closed points of $J_m(X/Y)$ not lying in the image of s_m .*

Proof. This is immediate from the corresponding maps on $\text{HS}_{B/A}^m$. \square

Definition 6.4. Let $X \rightarrow Y$ be a morphism of schemes and let $m \in \mathbb{Z}_{>0} \cup \{\infty\}$. Then the **Green-Griffiths projectivized jet bundle** is the scheme

$$P_m(X/Y) := \mathbf{Proj} \text{HS}_{X/Y}^m$$

over X .

Remark. If $m = 0$, then the above definition yields the empty scheme.

Example. When $m = 1$, $P_1(X/Y) \cong \mathbb{P}(\Omega_{X/Y})$, where $\mathbb{P}(\cdot)$ is the projective space of hyperplanes, as in ([EGA], II 4.1.1).

When $m > 1$, $\text{HS}_{X/Y}^m$ is not generated over \mathcal{O}_X by its elements of degree 1, so $\mathcal{O}(1)$ on $P_m(X/Y)$ is not a line sheaf. Instead, fibers of the structural morphism $\pi_m: P_m(X/Y) \rightarrow X$ are *weighted projective spaces* in the sense of Dolgachev [Do] (at

least when X is locally of finite type over Y). For finite m , $\mathcal{O}(d)$ is a line sheaf if d is divisible by all integers $1, \dots, m$.

As was the case with the earlier jet spaces, there are projection mappings

$$\pi_{ji}: P_j(X/Y) \dashrightarrow P_i(X/Y)$$

for all $1 \leq i \leq j \leq m$, and again they form an inverse system as in (4.8). They are only rational maps, though (unless $i = j$). There are also projections

$$\pi_i: P_i(X/Y) \rightarrow X$$

(which are morphisms). They commute with the π_{ji} : $\pi_i \circ \pi_{ji} = \pi_j$.

Remark 6.5. It is easy to see that the points of $P_m(X/Y) = \mathbf{Proj} \mathrm{HS}_{X/Y}^m$ are the nontrivial orbits of the action of \mathbb{G}_m , and that one can think of $P_i(X/Y)$ as consisting of certain \mathbb{G}_m -invariant prime ideals (as opposed to certain homogeneous prime ideals).

Proposition 6.6. *These projectivized jets spaces have the following properties.*

(a). (Functoriality) A commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{P_m(f)} & P_m(X'/Y') \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' , \end{array}$$

where U is the open subset of $P_m(X/Y)$ corresponding to the complement in $J_m(X/Y)$ of the pull-back via $J_m(f)$ of the image of $s_m: X' \rightarrow J_m(X'/Y')$.

(b). (Base change) In the situation of part (a), if $X = X' \times_{Y'} Y$, then $P_m(f)$ is defined everywhere on $P_m(X/Y)$ and induces an isomorphism

$$P_m(X/Y) \xrightarrow{\sim} P_m(X'/Y') \times_{X'} X$$

of schemes over X .

- (c). (Affine space) If $X \cong Y[x_i]_{i \in I}$, then $P_m(X/Y) \cong \mathbb{P}_X^{I,m}$, where $\mathbb{P}_X^{I,m}$ denotes $\mathbf{Proj} \mathcal{O}_X[x_{ij}]_{i \in I, j=1, \dots, m}$, with x_{ij} homogeneous of degree j for all i, j .
- (d). (Closed immersions) If a Y -morphism $f: X_1 \rightarrow X_2$ is a closed immersion, then $P_m(f/Y)$ is defined everywhere and is a closed immersion.

- (e). (Projectivity) If X is locally of finite type over Y and $m < \infty$, then $P_m(X/Y) \rightarrow X$ is projective (in the sense of ([EGA], II 5.5.2)).
- (f). (Étale morphisms) If a Y -morphism $f: X_1 \rightarrow X_2$ is étale, then $P_m(f/Y)$ is defined everywhere and induces an isomorphism

$$P_m(X_1/Y) \xrightarrow{\sim} P_m(X_2/Y) \times_{X_2} X_1 .$$

- (g). (Smooth morphisms) If the morphism $X \rightarrow Y$ is smooth of relative dimension d , then X is covered by open sets U for which $P_m(U/Y) \cong \mathbb{P}_U^{\{1, \dots, d\}, m}$. Moreover, this covering may be taken independent of m .

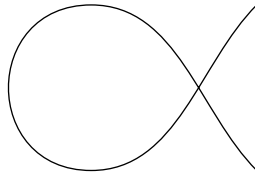
Proof. Left to the reader. □

§7. Logarithmic jets

As a general rule, whenever something is true in Nevanlinna theory for holomorphic maps to complex manifolds X , its (appropriately stated) counterpart for holomorphic maps to complex manifolds X relative to a normal crossings divisor D also holds. For this reason, it would be useful to define jets in the above context. For normal crossings divisors on complex manifolds, the appropriate notion is jets with logarithmic poles along the divisor (e.g., $d \log f$, $d^2 \log f$, etc., where f is a local defining equation for D). This concept was introduced by Noguchi [N].

I believe that this concept of log jets can also be defined in the present context, of algebraic geometry, with arbitrary singularities allowed, but this has not been checked, for the following reasons.

The key problem is that, when working with schemes, one uses the Zariski topology, which is much coarser than the (classical) topology on a complex manifold. As an example, consider an irreducible nodal curve on a smooth surface over a field.



In the complex topology, it is easy to find an open neighborhood of the node in which the two branches of the curve are defined by different equations. In the Zariski topology, however, this is not the case: any open neighborhood of the intersection point will contain all but finitely many points of the curve, and therefore the curve will always remain irreducible. This problem is traditionally solved by working locally in the étale topology; the key result that enables one to do this seems to be the following:¹

¹This seems to be so well known that nobody has a reference for it. Does anyone know of a convenient reference?

Theorem 7.1. *Let $f: X \rightarrow Y$ be a closed immersion of locally noetherian excellent schemes, let $y \in Y$ be a point, let \mathcal{O}_y denote the local ring at y , and let Z be an irreducible component of $X \times_Y \operatorname{Spec} \widehat{\mathcal{O}}_y$. Then there is an étale map $Y' \rightarrow Y$, a point $y' \in Y'$ lying over y with the same residue field $k(y') = k(y)$ (and therefore $\widehat{\mathcal{O}}_{y'} \cong \widehat{\mathcal{O}}_y$), and an irreducible component Z' of $X' := X \times_Y Y'$, such that*

$$(7.1.1) \quad Z' \times_{Y'} \operatorname{Spec} \widehat{\mathcal{O}}_{y'} = Z.$$

Proof. We start with some commutative algebra. Let A be an excellent local ring with maximal ideal \mathfrak{m} , and let A^h denote its henselization. By ([EGA], IV 18.6.6 and 18.7.6), A^h is an excellent local ring, and the structural map $A \rightarrow A^h$ is a local homomorphism. In addition, by ([EGA], IV 18.5.14 and 18.6.6), the structural map $A \rightarrow \widehat{A}$ factors through A^h , and also $\widehat{A^h} = \widehat{A}$. If I is an ideal of A , then by ([EGA], IV 18.6.8), $(A/I)^h = A^h/IA^h$. Combining this with standard properties of completions, it follows that the diagram

$$\begin{array}{ccccc} A/I & \longrightarrow & (A/I)^h & \longrightarrow & \widehat{A/I} \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A^h & \longrightarrow & \widehat{A} \end{array}$$

is Cartesian. Finally, by ([EGA], proof of IV 18.6.12), $\sqrt{IA^h} = \sqrt{I}A^h$.

Now let A be the local ring \mathcal{O}_y , let $I \subseteq A$ be the ideal corresponding to X , and let $\sqrt{I}A^h = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a minimal primary decomposition of $\sqrt{I}A^h = \sqrt{I}A^h$ in A^h . It is easy to check that the \mathfrak{q}_i must in fact be prime. By ([EGA], IV 18.9.2), $\mathfrak{q}_i \widehat{A}$ is also prime for all i . By ([E], Thm. 7.2b) and ([M], Thm. 7.4(ii)),

$$\sqrt{I}\widehat{A} = \mathfrak{q}_1 \widehat{A} \cap \cdots \cap \mathfrak{q}_r \widehat{A};$$

therefore some subset of the $\mathfrak{q}_i \widehat{A}$ gives a primary decomposition of $\sqrt{I}\widehat{A}$. In particular, Z comes from one of these prime ideals, which we may assume to be \mathfrak{q}_1 .

By ([EGA], IV 18.6.5), A^h is the inductive limit of all **strictly essentially étale** A -algebras B ; by definition these are local homomorphisms $A \rightarrow B$ of local rings such that the induced map of residue fields is an isomorphism and such that there is an étale A -algebra C and an ideal \mathfrak{n} of C lying over \mathfrak{m} such that $C_{\mathfrak{n}} \cong B$ as A -algebras.

Let B be a strictly essentially étale A -algebra containing (finite) generating sets for each of the \mathfrak{q}_i ; then there are ideals \mathfrak{q}'_i in B such that $\mathfrak{q}'_i A^h = \mathfrak{q}_i$ for all i . By ([EGA], IV 18.6.5), $B^h = A^h$; hence by ([EGA], IV 18.6.6), A^h is faithfully flat over B . Thus $\sqrt{I}B = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_r$. This remains true if we replace \mathfrak{q}'_i with the prime ideal $\mathfrak{q}'_i A^h \cap B$ for all i .

Now let C be an étale A -algebra and let \mathfrak{n} be a prime ideal of C such that $C_{\mathfrak{n}} \cong B$; then there is a prime ideal \mathfrak{p} of C corresponding to \mathfrak{q}_1 . The generators and

relations for C give a scheme Y' of finite type over some open affine in Y ; let y' be the point of Y' corresponding to \mathfrak{n} and let Z' be the closed subset corresponding to \mathfrak{p} . Then, after replacing Y' with a suitable open subset ([EGA], IV 17.6.1), it satisfies the conditions of the theorem. \square

In the slightly less general setting where Y is a scheme of finite type over a field or over an excellent Dedekind domain, this can also be handled by applying ([A], Cor. 2.1), using the radical of a product of ideals.

Definition 7.2. Let X be a regular locally noetherian excellent scheme. Then a **normal crossings divisor** on X is an effective divisor D on X such that, locally in the étale topology, irreducible components of the support are regular and cross transversally. If this is the case, then we also say that D **has normal crossings**.

Theorem 7.1 suggests that $\mathrm{HS}_{X/Y}^m(\log D)$ should be defined not as a sheaf over X in the usual sense, but as an étale sheaf; this is necessary so that $\mathrm{HS}_{X/Y}^m(\log D)$ will have the expected properties when Y is a field, X is a smooth scheme over Y , and D is a normal crossings divisor on X . Under suitably nice circumstances, this will then descend to a quasi-coherent sheaf on the Zariski topology (with a non-obvious definition).

In order to obtain more generality, though, it would be better to work in the theory of logarithmic schemes. For more details on log schemes, see [K].

This goes far beyond the scope of the present note, so the case of log jets will be left for future work.

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